Altorientalische Forschungen	20	1993	2	245–260

JENS HØYRUP

Mathematical Susa Texts VII and VIII. A Reinterpretation

Burchard Brentjes und Ernst Werner
gewidmet

In an extensive paper published due to the kindness of the editorial staff of the present journal, I have suggested a geometrical reinterpretation of so-called Old Babylonian "algebra".[1] Among the texts analyzed in the paper were the mathematical texts No. XVI and No. IX from Susa, which turned out to contain highly illuminating didactical commentaries of a kind not known from the Babylonian core area – be it because teaching in a peripheral area felt a need to make explicit what could be left to a stable oral tradition in the core, or simply because the Susa teachers had a bent for loquacity.

Text No. XVI turned out not to contain solutions of problems but only a didactical discussion of transformations of linear "equations" of two unknowns (as usually, the uš ("length") and ("width") of a rectangle, with the usual values 30′ and 20′ [nindan][2]). No. IX contained initial didactical discussions of the transformations of complex into simpler second-degree "equations" followed by use of the technique just taught for the solution of a sophisticated set of equations.

These texts are not the only Susa texts to contain illuminating didactical commentaries. In the present paper I shall analyze two further texts, of which one contains an explicit explanatory part, while the other employs some of the concepts introduced in the former.

An extra reason for reanalyzing the two texts is that the treatment given by

[1] J. Høyrup, Algebra and Naive Geometry. An Investigation of Some Basic Aspects of Old Babylonian Mathematical Thought, in: AoF 17 [1990], 27–69, 262–354. – Abbreviations: MKT – O. Neugebauer, Mathematische Keilschrift-Texte, I–III, Berlin 1935, 1935, 1937 (Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik. Abteilung A: Quellen. 3. Bd., 1.–3. Teil); TMS – E. M. Bruins – M. Rutten, Textes mathématiques de Suse, Paris 1961 (Mémoires de la Mission Archéologique en Iran, XXXIV).

[2] I shall use F. Thureau-Dangin's system for translating sexagesimal place value numbers: `, ``, etc. designate increasing and ′, ″ etc. decreasing sexagesimal orders of magnitude. When needed, ° is used to indicate „order zero" (1° = 1).

E. M. Bruins in the original edition (TMS, 52–62) was highly unsatisfactory even in terms of the received arithmetical interpretation of such texts. In the first text he misread a possessive suffix -*šu* for a Sumerian ŠU, which made him invent a specific "heuristic method of the hand", which has since then spread into the secondary literature[3], and made him mistake one indeterminate equation, something very rare in Babylonian mathematics, for a trivial set of two first-degree equations. In the second text, a number of translations and repairs to the text are overly fanciful, while obvious restitutions suggested by parallel passages are left out.

TMS VII: Indeterminate first-degree problems

Let us first look at Text VII, which runs as follows:[4]

Problem A

1. 4-*at* sag *a-na* uš daḫ 7-⟨*ti-*⟩*šu*[5] *a-na* 10 [*al-li-ik*]
The 4th of the width to the length I have appended, its 7⟨th⟩, to 10 [I have gone,]

2. *ki*!-*ma* UL.GAR uš *ù* ⟨sag⟩ za-e 4 gar 7 [gar]
as much as the accumulation of length and ⟨width⟩. You, 4 pose; 7 [pose;]

3. 10 gar 5 *a-na* 7 *i-ší* 35 *ta-mar*
10 pose; 5′ to 7 raise, 35′ you see.

4. 30 *ù* 5 *be-e-er* 5 a-rá! *a-na* 10 *i-ší*
30′ and 5′ single out. 5′, the step, to 10 raise,

5. 50 *ta-mar* 30 *ù* 20 gar 5 a-rá! *a-na* 4 *re-*⟨*ba-ti*⟩ sag
50′ you see. 30′ and 20′, pose. 5′, the step, to 4, of the four⟨th⟩ of the width,

6. *i-ší-ma* 20 *ta-mar* 20 sag 30 *a-na* 4 *re-ba-*⟨*ti*⟩
raise: 20′ you see, 20′, the width. 30′ to 4, of the fourt⟨h⟩,

[3] So, H. Gericke (Mathematik in Antike und Orient, Berlin etc. 1984) borrows E. M. Bruins' interpretation of problem VII A as his first example of Babylonian algebra (pp. 25–32).

[4] See, apart from the transliteration in TMS, 52ff., the autography (plates 14f.), and W. von Soden's review of TMS, in: BiOr 21 [1964] 44–50, here 48.

[5] This -*šu* is what becomes a „hand" in E. M. Bruins' interpretation. That a -*ti*- has simply been omitted and a 7th is thus meant can be seen from a number of parallel passages (so line 17 of the same tablet; TMS IX, line 20; VAT 8520, obv. 1, rev. 4, in MKT I, 346f.). The reading is confirmed by the consistency of the text which is obtained.

7. *i-ší* 2 *ta-mar* 2 gar uš 20 *i-na* 20 zi
raise, 2 you see. 2 pose, lengths. 20′ from 20′ tear out,

8. *ù i-na* 2 30[6] zi 1,30 *ta-mar*
and from 2, 30′ tear out, 1°30′ you see.

9. *i-na* 4 *re-ba-ti* 1 zi 3,20[7] *ta-mar*
From 4, of the fourth, 1 tear out, 3 {+ 20′} you see.

10. igi 3 *pu-ṭú-*⟨*úr*⟩ 20 *ta-mar* 20 *a-na* 1,30 *i-ší-ma*
The igi of 3 deta⟨ch⟩, 20′ you see. 20′ to 1°30′ raise:

11. 30 *ta-mar* 30 uš 30 *i-na* 50 zi 20 *ta-mar* 20 sag
30′ you see, 30′ the length. 30′ from 50′ tear out, 20′ you see, 20′ the width.

12. *tu-úr* 7 *a-na* 4 *re-ba-*⟨*ti*⟩ *i-ší* 28 *ta-mar*
Turn back. 7 to 4, of the fourt⟨h⟩, raise, 28 you see.

13. 10 *i-na* 28 zi 18 *ta-mar* igi 3 *pu-*⟨*ṭú-úr*⟩
10 from 28 tear out, 18 you see. The igi of 3 det⟨ach⟩,

14. 20 *ta-*⟨*mar*⟩ 20 *a-na* 18 *i-ší* 6 *ta-mar* 6 uš
20′ you s⟨ee⟩. 20′ to 18 raise, 6 you see, 6 (for) the length.

15. 6 *i-na* 10 zi 4 sag 5 *a-na* 6 [*i-š*]*í*
6 from 10 tear out, 4 (for) the width. 5′ to 6 [raise,]

16. 30 uš 5 *a-na* 4 *i-ší* 20 *ta-*⟨*mar*⟩20 ⟨sag⟩
30′ the length. 5′ to 4 raise, 20′ you s⟨ee⟩, 20′ the ⟨width⟩.

Problem B

17. 4-*at* sag *a-na* uš daḫ 7-*ti*[-*šu*]
The fourth of the width to the length I have appended, [its] 7th

18. *a-di* 11 *al-li-ik* ugu! [UL.GAR]
until 11 I have gone, over the [accumulation]

[6] It appears from the autography, either that the two numbers were at first written together but 30 then deleted and rewritten with distance; or that some small wedges marking a separation are written between the two numbers.

[7] As observed by E. M. Bruins, the scribe has tried to correct this number (which should be 3) but has done so incorrectly. 3°20′ will have been on the scribe's mind as 4. (length + width).

19. uš *ù* sag 5 dirig za-e [4 gar]
of length and width 5′ it goes beyond. You, [4 pose;]

20. 7 gar 11 gar *ù* 5 gar
7 pose; 11 pose; and 5′ pose.

21. 5 *a-na* 7 *i-sí* 3[5 *ta-mar*]
5′ to 7 raise, 3[5′ you see.]

22. 30 *ù* 5 gar 5 *a-na* 1[1 *i-ší* 55 *ta-mar*]
30′ and 5′ pose. 5′ to 1[1 raise, 55′ you see.]

23. 30 20 *ù* 5 zi gar 5 [*a-n*]*a* 4
30′, 20′ and 5′, to tear out, pose. 5′ [t]o 4

24. *i-ší* 20 *ta-*⟨*mar*⟩ 20 sag 30 *a-na* 4 *i-ší-ma*
raise, 20′ you s⟨ee⟩, 20′ the width. 30′ to 4 raise,

25. 2 *ta-mar* 2 uš 20 *i-na* 20 zi
2 you see, 2, lengths. 20′ from 20′ tear out.

26. 30 *i-na* 2 zi 1,30 gar *ù* 5 *a-*[*na* ...]
30′ from 2 tear out, 1°30′ pose, and 5′ t[o ...]

27. 7 *a-na* 4 *re-*⟨*ba-ti*⟩ *i-ší-ma* 28 *ta-mar*
7 to 4, of the four⟨th⟩, raise, 28 you see.

28. 11 UL.GAR *i-na* 28 zi 17 *ta-mar*
11, the accumulations, from 28 tear out, 17 you see.

29. *i-na* 4 *re-*⟨*ba-ti*⟩ 1 zi 3 [*ta*]*-mar*
From 4, of the four⟨th⟩, 1 tear out, 3 y[ou] see.

30. igi 3 *pu-ṭú-*⟨*úr*⟩ 20 *ta-*⟨*mar*⟩ 20 [*a-na*] 17 *i-*⟨*ší*⟩
The igi of 3 deta⟨ch⟩, 20′ you s⟨ee⟩. 20′ t[o] 17 ra⟨ise⟩,

31. 5,40 *ta-*⟨*mar*⟩ 5,40 [u]š 20 *a-na* 5 dirig *i-ší*
5°40′ you s⟨ee⟩, 5°40′, (for) the [le]ngth. 20′ to 5′, the going-beyond, raise,

32. 1,40 *ta-*⟨*mar*⟩ 1,40 *wa-ṣí-ib* uš 5,40 uš
1′40″ you s⟨ee⟩, 1′40″, the appending of the length. 5°40′, (for) the length,

33. *i-na* 11 UL.GAR zi 5,20 *ta-mar*
from 11, accumulations, tear out, 5°20′ you see.

34. 1,40 *a-na* 5 dirig daḫ 6,40 *ta-mar*
1′40″ to 5′, the going-beyond, append, 6′40″ you see.

35. 6,40 *n*[*a*]*-sí-iḫ* sag 5 a-rá
6′40″, the t[ea]ring-out of the width. 5′, the step,

36. *a-na* 5,40 uš *i-ší* 28,20 *ta-mar*
to 5°40′, lengths, raise, 28′20″ you see.

37. 1,40 *wa-sí-ib* uš *a-na* 28,20 [daḫ]
1′40″, the appending of the length, to 28′20″ [appe]nd,

38. 30 *ta-mar* 30 uš 5 *a-*[*na* 5,20]
30′ you see, 30′ the length. 5′ t[o 5°20′]

39. *i-ší-ma* 26,40 *t*[*a-mar* 6,40]
raise, 26′40″ yo[u see. 6′40″,]

40. *na-sí-iḫ* sag *i-na* [26,40 zi]
the tearing-out of the width, from [26′40″ you tear out,]

41. 20 *ta-mar* 20 sa[g] (...?)
20′ you see, 20′ the wid[th.]

First of all, the terminology must be explained briefly (for more detailed discussions, see Høyrup, AoF 17,45–65):
1) *waṣābum*/daḫ (translation "to append"[8]) is an asymmetric additive process, in which one quantity is joined to another of the same kind. The latter can thus be said to conserve its identity (and, in geometric manipulation, its place) while being enlarged, while the former is absorbed (and moved if necessary in cases of geometric manipulation). Since all concrete entities are assumed to possess measuring numbers, the operation entails an arithmetical addition (similarly for all concrete processes discussed in what follows). From *waṣābum* originates the term *waṣībum* used here and elsewhere in the Susa corpus, "that which is to be appended", for convenience translated "the appending".
2) *nasāḫum*/zi ("to tear out.") is the corresponding subtractive process, by which a part of an entity is removed. Whence *nasīḫum,* "that which is to be torn out", for convenience "the tearing-out".

[8] I follow a principle of „conformal translation", where the same term is always translated in the same way, different terms (unless logographic equivalence is established beyond doubt within the text group) translated differently, and terms derived from the same root as far as possible rendered by similarly related translations. The result is clumsy but allows that the actual operations of the texts can be discussed in (some kind of) English.
An extensive list of suggested standard translations is given in Høyrup, AoF 17, 67–69.

3) *kamārum*/gar-gar/UL.GAR ("to accumulate") is a symmetric additive process. It may be a genuine arithmetical operation by which measuring numbers of entities of different kinds (e.g., areas and lengths) are added; but it may probably also be meant as a concrete putting-together. UL.GAR may also disignate the sum by this addition ("the accumulation")
4) The phrase A *eli* BD *watārum*/A ugu BD dirig ("A over BD goes beyond") is an operation by which two entities A and B are compared. Since $D = A-B$, we may speak of a "subtraction by comparison". The difference D may be spoken of as dirig (translated "going-beyond").
5) *našûm*/il ("to raise") is a multiplicative process, designating the calculation of a concrete magnitude by multiplication.
6) In the present text, *alākum* ("to go") appears as a multiplication; in TMS VIII, however, it stands for repeated appending of the same entity. The basic idea is thus the repetition of a certain step, and context will have to tell the precise meaning. The corresponding verbal noun *tālukum* ("the going") stands for the total distance gone.
7) a-rá ("times") is the term used in multiplication tables. It is thus the arithmetical multiplication of number by number. In the present text it is used as a noun, referring to the line segment which is "gone". Since rá is the Sumerian equivalent of *alākum* it is translated "step".[9]
7) igi n is the reciprocal of n as found in reciprocal tables. Finding the igi is termed *paṭārum*/du$_8$ ("to detach").
8) *šakānum*/gar ("to pose") designates (apparently a number of different) processes of material recording, most probably in writing and drawing, perhaps also in a calculational device.

The rest of the terminology with appurtenant translations should be more or less self-explanatory, and we may thus start the analysis of what goes on in problem A.

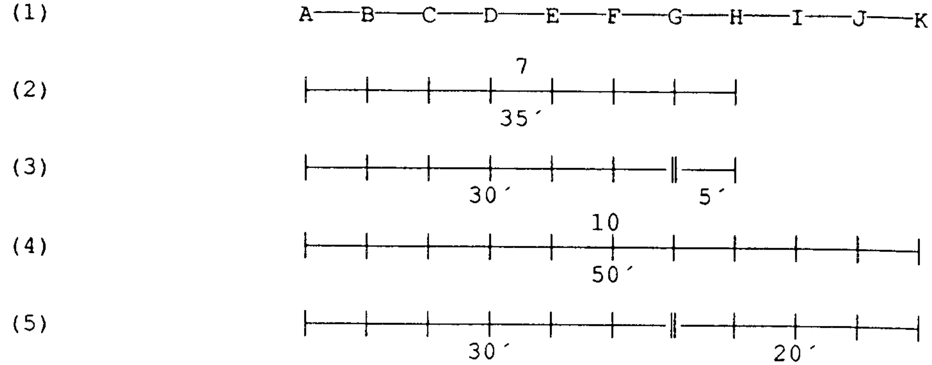
Formally, it deals with the "length" uš and the "width" sag of a rectangle. In the present problem, however, the terms are just labels for two line segments. It may be easier to follow the procedure if we allow ourselves the anachronism of a translation into algebraic symbolism, writing x for the length and y for the width – but it is important that this translation is only used as a support, since it abolishes, e.g., the distinction between different additive operations.

Lines 1–2 tell that a 4th of the width has been appended to the length, that the 7th of the outcome has been taken 10 times, amounting in total to the accumulation of length and width. In symbols:

$$\tfrac{1}{7}(x + \tfrac{1}{4}y)\cdot 10 = x + y$$

Lines 2–5 now explain the meaning of this equation by extensive "posing", which we may imagine to take place in a diagram like this:

[9] This corresponds to the usage of the Seleucid text BM 34568–see Høyrup, AoF 17, 343f.



First the numbers 4, 7 and 10 – divisors and multiplier – are recorded. Then the magnitude 5′ (the "step" of line 5 – taken at this point to be known) is raised to 7, giving 35′, which can be decomposed as 30′ + 5′ ($x + \frac{1}{4}y$). Next it is raised to 10, which gives 50′, decomposable into 30′ + 20′ ($x + y$). It may seem astonishing to us that the meaning of an equation is discussed in terms of its solution; but the method is attested elsewhere in Old Babylonian mathematical texts (e.g., TMS IX and XVI), and it is a quite effective didactical substitute for algebraic symbols.

So far the text has presented us with a didactical exposition of the numerical foundation of the original equation. Lines 5–11 go on with a similar exposition of the meaning of the first step of the transformation of the equation, which is a multiplication by 4,

$$\tfrac{1}{7}[4x + y]\cdot 10 = 4\cdot(x + y).$$

We look at the decomposition 35′ = 30′ + 5′ ((3) in the diagram): multiplying 5′ by 4 gives 20′, the width; multiplying 30′ yields 3, (4) lengths. Now one width and one length are removed. Tearing out a width (20′) from 20′ leaves, literally, n o t h i n g w o r t h s p e a k i n g a b o u t: tearing out a length (30′) from the 4 lengths (2) leaves 1°30′; this is found in line 9 to correspond to 3 (*viz* lengths). Multiplying by $\frac{1}{3}$ = 20′ indeed gives 30′, the length; tearing this out from 50′ leaves 20′, the width.

The *tu-úr,* "turn back", of line 13 tells that the explanations are now finished and the procedure is to begin. After the multiplication by 4 which was already explained, the equation is multiplied by 7, which leads from

$$\tfrac{1}{7}[(4-1)\ x + (x + y)]\cdot 10 = 4\cdot(x + y)$$

to

$$3x\cdot 10 + (x + y)\cdot 10 = 28\cdot(x + y)$$

and hence to

$$3x \cdot 10 = 18 \cdot (x+y)$$

Dividing this by 3 (raising to igi $3 = 20'$) yields

$$x \cdot 10 = 6 \cdot (x+y).^{10} \qquad (*)$$

The easiest solution to this indeterminate equation is found if we put the first factors equal, $x = 6$, and the second factors equal, $10 = x + y$, from which of course y can be found as $10 - 6$. This seems in fact to be what happens, either on this arithmetical level or perhaps, as it may be intimated by problem B, by imagining the factors as sides of a rectangle (see Figure 1).

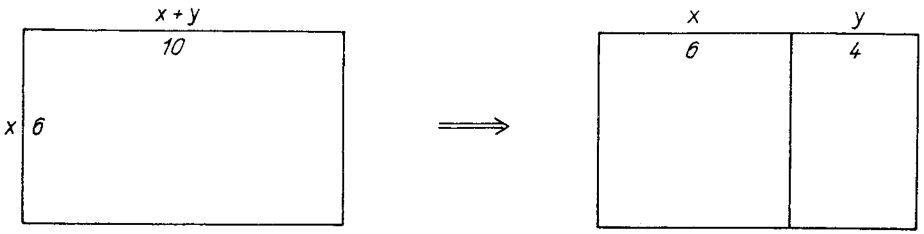


Fig. 1

Evidently, this is not the only set of solutions to the equation (whence the "(for)" in the translation); in fact, every set $6z$, $4z$ will do. By multiplying by $5'$ the text obtains the solution presupposed in the beginning, $x = 30'$, $y = 20'$.

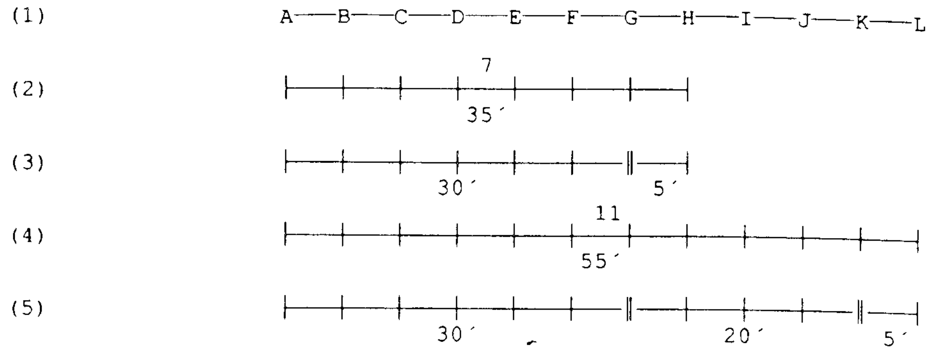
The $5'$ used here is the "step" from line 4, as told in the parallel passage in line 35. There is no compelling reason that precisely this factor should transform a solution obtained from (*) in the way indicated into the set originally thought of – unless the division leading to (*) is chosen so as to give a preliminary value equal to the number of "steps" going into the intended length. We must therefore assume that this is precisely what was done. This would also explain why the equation is not reduced eben further, *viz* to $x \cdot 5 = 3 \cdot (x+y)$.

Problem B, again, deals with length and width of a rectangle, and takes the same 7th. This time, however, 11 steps are made, which makes us go $5'$ beyond the accumulation of length and width:

$$\tfrac{1}{7}(x + \tfrac{1}{4}y) \cdot 11 = x + y + 5'.$$

Again, we may refer the "posings" of lines 19 to 23 to a diagram:

[10] Evidently, the representations of the equations will have been wholly different. They may have been rhetorical (verbal). More likely, however, is something in the likeness of the diagram above, or a kind of number scheme–cf. the discussion of schematic and graphic representations of TMS XVI in Høyrup, AoF 17, 305.



Lines 23 to 26, again, explain the multiplication of $(x + \tfrac{1}{4}y)$ by 4 and the ensuing transformation. The end of line 26 suggests that a further transformation corresponding to

$$\tfrac{1}{7}[(4-1)\,x - 5' + (x + y + 5')] \cdot 11 = 4 \cdot (x + y + 5')$$

is prepared, but a damage to the tablet prevents us from knowing precisely how.

In line 27, the procedure starts for good. We may follow it in symbols, from

$$\tfrac{1}{7}[(4-1)x - 5' + (x + y + 5')] \cdot 11 = 4 \cdot (x + y + 5')$$

to

$$11 \cdot [3(x - \tfrac{1}{3} \cdot 5')] + (x + y + 5') \cdot 11 = 28 \cdot (x + y + 5')$$

and hence to

$$11 \cdot (x - 1'40'') = 5°40' \cdot (x + y + 5').$$

Already at this stage, $5°40'$ is ascribed to the length. The entity to which it corresponds is, however, not the original length x but $x' = x - 1'40''$, where $1'40''$ is explicitly introduced as the "appending of the length".[11]

Concomitantly, 11 will have to correspond to $x + y + 5'$. Tearing out $5°40'$ for x' leaves $5°20'$ to correspond to a y', which in line 34 is found to be $(x + y + 5) - (x - 1'40'') = y + (5' + 1'40'') = y + 6'40''$, where $6'40''$ is called the "tearing-out of the width". The ease by which this computation is carried out corroborates the conjecture that a geometric or similar intuitively transparent representation is used (cf. Figure 2). In any case, the possible solutions

[11] The construct state *waṣīb* demonstrates that we really have to do with a noun, while its role in the calculation shows that it is of gerundive type. In German it would be called „das Hinzuzufügende".

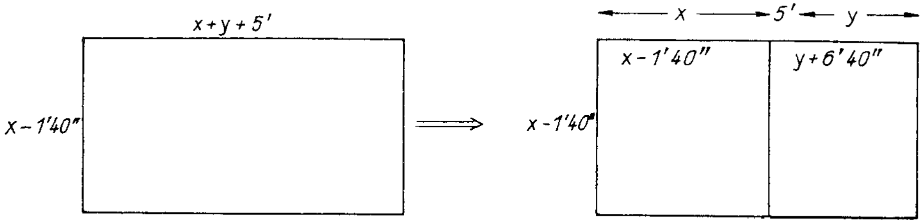


Fig. 2

$x'=5^\circ 40', y'=5^\circ 20'$ are multiplied by the step 5′, giving the preferred solutions $x'=28'20'', y'=26'40''$. Appending and tearing out what according to their names should be appended and torn out finally gives $x=30', y=20'$.

After its didactical introduction, problem A thus demonstrates how to solve a homogeneous indeterminate first-degree problem. Problem B, on its part, shows how to reduce an inhomogeneous to a homogeneous problem through a shift of variables made very explicit by the "appending" and "tearing-out".

Obviously, an equation $a\cdot u=b\cdot v+c$ can be brought much more easily on homogeneous form by means of the principles used in our text – *viz* as $a\cdot u=b\cdot(v+{}^{c}/_{b})$. The method which is actually followed, which reduces the problem into one in x and $x+y$ (or x' and $x'+y'$), shows that the so-called "accumulation" (UL.GAR, $x+y+5'=x'+y'$) was regarded as an entity of its own; it even seems to betray that this entity was regarded as fundamental, not just introduced *ad hoc* for the construction or solution of certain problems.

TMS VIII: Mixed second-degree problems

The same entities uš, sag, *waṣībum* and *nasīḫum* are also made use of in Text VIII, which otherwise deals with a quite different problem type. The text looks as follows:

Problem A

1. [a-šà 10 4-*at* sag *a-na* sag daḫ] *a-na* 3 *a-li-ik* ... ... [ugu]
   [The surface 10′. The 4th of the width to the width I have appended,] until 3 I have gone ... over

2. [uš 5 dir]ig za.e [4 *r*]*e-ba-ti ki-ma* sag gar *re-b*[*a-at* 4 *le-qé* 1 *ta-mar*]
   [the length 5′ goes] beyond. You, [4, of the f]ourth, as much as width pose. The four[th of 4 take, 1 you see.]

3. [1 *a-na*] 3 *a-li-ik* 3 *ta-mar* 4 *re-ba-at* sag *a-na* 3 d[aḫ 7 *ta-mar*]
   [1 to] 3 go, 3 you see. 4 fourths of the width to 3 ap[pend, 7 you see.]

4. [7] *ki-ma* uš gar 5 dirig *a-na na-si-iḫ* uš gar 7 uš *a-na* 4 [*i-ši*]
   [7] as much as length pose. 5′ the going-beyond to the tearing-out of the length pose. 7, of the length, to 4 [raise,]

5. 28 *ta-mar* 28 a-šà 28 *a-na* 10 a-šà *i-ši* 4,40 *ta-mar*
   28 you see. 28, of the surfaces, to 10′ the surface raise, 4°40′ you see.

6. [5] *na-si-iḫ* uš *a-na* 4 sag *i-ši* 20 *ta-mar* $^1/_2$ *ḫe-pe* 10 *ta-mar* NIGIN
   [5′], the tearing-out of the length to four, of the width, raise, 20′ you see. $^1/_2$ break, 10′ you see. Make surround,

7. [1,40] *ta-mar* 1,40 *a-na* 4,40 daḫ 4,41,40 *ta-mar mi-na* íb-si 2,10 *ta-ma*[*r*]
   [1′40″] you see. 1′40″ to 4°40′ append, 4°41′40″ you se[e.] How much the equilateral? 2°10′ you see.

8. [10 ¿S]Á!SÁ?*a-na* 2,10 daḫ 2,20 *ta-mar mi-na a-na* 28 a-šà gar *ša* 2,20 *i-na-*[*di-n*]*a*
   [10 to the e]qual(?) to 2°10′ append, 2°20′ you see. How much to 28, of the surfaces, shall I pose which 2°20′ gi[ve]s me?

9. [5 gar] 5 *a-na* 7 *i-ši* 35 *ta-mar* 5 *na-si-iḫ* uš *i-na* 35 zi
   [5′ pose.] 5′ to 7 raise, 35′ you see. 5′, the tearing-out of the length from 35′ tear out,

10. [30 *ta-*]*mar* 30 uš 5 uš *a-na* 4 sag *i-ši* 20 *ta-mar* 20 {uš} ⟨sag⟩
    [30′ you]see, 30′ the length. 5′ the length[12] to 4 of the width raise, 20′ you see, 20 the length (mistake for width).

Problem B

11. [a-šà 10] 4-*at* sag *a-na* uš daḫ *a-na* 1 *a-li-ik a-na* ¿KI/DI [u]š ugu? sag ¿*i-ši-ma*?
    [The surface 10′.] The 4th of the width to the length (probably erroneously for width[13]) append, to 1 go, (the outcome falls 5′ short of the length[14])

[12] This probably refers to the „length" of the square $^y/_4\cdot{}^y/_4$. Several other mathematical Susa texts (N^os^ V and VI), indeed, speak about the „length" of a square.

[13] Unless the meaning (or the reason for this slip of the stylus) is „append ⟨to the equivalent of the width⟩ along the ⟨direction of the⟩ length.

[14] As pointed out by W. von Soden (BiOr 21, 48a), a better interpretation of this (so far nonsensical) passage can only be attained through collation. Since neither the gar nor the *le-qé* in the next line look as they should on the autography, the first lines of the reverse (line 11–12) must presumably be quite damaged (TMS contains no photo of the tablet, and it has only reappeared quite recently afterhaving been mislaid in the Louvre collection for decades–Jim Ritter, personal communication).

12. [za.]e 4 *re-ba-ti ki-ma* sag gar *re-ba-at* 4 *le-qé* 1 *ta-mar* 1 *a-na* 1 *a-l*[*i-ik*]
[Yo]u, 4, of the fourth, as much as width pose. The fourth of 4 take, 1 you see. To 1 g[o.]

13. [¿1?] 4 gaba 4 gar 1 *ta-lu-ka a-na* 4 daḫ 5 *ta-*⟨*mar*⟩ *ki-ma* uš gar
[1(?)] 4 the counterpart, 4 pose. 1, the going, to 4 append, 5 you s⟨ee⟩, as much as length pose.

14. [5] *wa-ṣi-ib* uš gar 5 uš *a-na* 4 sag *i-ši* 20 *ta-mar* 20 a-šà
[5′,] the appending of the length, pose. 5, of the length, to 4, of the width, 20 surfaces.

15. [20 *a-na* 10] [*i-ši*] 3,20 *ta-mar* 5 *wa-ṣi-ib* ⟨uš⟩ *a-na* 4 sag *i-ši* 20 *ta-mar*
[20 to 10′ r]a[ise], 3°20′ you see. 5′, the appending of ⟨the length⟩, to 4 raise, 20′ you see.

16. [$^1/_2$ *ḫe-pe* 10 *ta-mar*] NIGIN 1,40 *ta-mar* 1,40 *a-na* 3,20 daḫ 3,21,40 *ta-mar*
[$^1/_2$ break, 10′ you see.] Make surround, 1′40″ you see. 1′40″ to 3°20′ append, 3°21′40″ you see.

17. [*mi-na* íb-si 1,50 *ta-mar* 10] ¿SÁ!SÁ!?[15] *i-na* 1,50 zi 1,40 *ta-mar*
[How much the equilateral? 1°50′ you see. 10′] the equal(?), from 1°50′ tear out, 1°40′ you see.

18. [igi 20 a-šà *pu-ṭú-úr* 3 *ta-mar* 3] *a-na* 1,40 *i-ši* 5 *a-na* 5 uš
[The igi of 20, of the surfaces, detach, 3′ you see. 3′] to 1°40′ raise, 5′ to 5, of the length.

19. [*i-ši* 25 *ta-mar* 5 uš *wa-ṣi-ib a*]*-na* 25 daḫ 30 *ta-mar* 30 uš
[raise, 25′ you see. 5′, the appending of the length, t]o 25′ append, 30′ you see, 30′ the length.

20. ... 20 sag
... 20, the width

Problem C

21. [... ... ...*ki*]*-ma* UL.GAR 3 uš *ù* 4 sag
[... ... ... as m]uch as the accumulation of 3 lengths and 4 widths

15 Bruins and Rutten read the first signs in line 8 as *a-di,* and the analogous group here as *le-qé,* none of which make sense. The autography agrees acceptably well with the assumption that the same signs are used in both cases. The second sign appears te be a DI; the first is too close to the breaks to be read with certainty, but might be another DI. Though no mathematical standard expression, this makes sense if read SÁ.SÁ ∽ *šāninum,* "that which is equal", the current conceptualization of the side of a square.

22. [... ... ...*t*]*a-mar*
[... ... ... yo]u see

Once again, a few terms have to be explained.
1) *ḫepûm*/gaz ("to break") is the procedure by which (among other things) the "coefficient of the first-degree term" of a second-degree problem is bisected (the real geometrical meaning will be clear below).
2) NIGIN (an approximate ideographic but hardly a logographic equivalent of *šutamḫurum;* here translated "to make surround") is the construction process by which a line and its "counterpart" are made the sides of a square, thus "surrounding" it SÁ.SÁ ("the equal") in lines 8 and 16 appears to designate this line.
3) *meḫrum*/gaba is the "counterpart", i.e., in mathematical texts the "other side" of a square–mostly (but not here) it is used about the squares which result from quadratic completions.
4) a-šà (never fully written in Akkadian but at times with the phonetic complements of *eqlum;* translated "surface") designates primarily the concrete extension of geometric figures; as always, however, these are presupposed to have a measuring number (an area).
Like the previous problems, those of Text VIII deal with the length and width of a rectangle. This time, however, the rectangle is real, since it possesses a surface 10′ (in both problem A and B; problem C will presumably have dealt with the same rectangle, but too little of it is conserved to prove this). This may make us suspect that the sides, once again, are $x = 30'$ and $y = 20'$. However, the present problems contain no introductory didactical explanation, so we may be supposed not to know.

In problem A, apart from the surface, we are told that 3 4ths of the width appended to the width exceeds the length by 5′–in symbolical translation

$$y + 3 \cdot {}^1/_4\, y = x + 5'.$$

In line 2 we are told to pose 4 "as much as the width". The "4 fourths" (in absolute state) in line 3 shows that this means taking $z = {}^1/_4\, y$ as a new unit. Returning to line 2, we see that one 4th (of the width) is then 1 [z]. Repeating it 3 times gives 3, for which reason the length including a tearing-out of 5′ will be $7z$ (line 4).

Lines 4–5 calculates the number of surfaces $z{\cdot}z$ contained by $4z$ times $7z$ to be "28 surfaces". That this, and not "28 the surface", is how the expression is to be read, follows from the multiplication which is used. If "a surface 28" was found, the multiplication would have been a construction, presumably *šutākulum*/ í-kú-kú (but possibly NIGIN, UL.UL or UR.UR.). "Raising" presupposes that a geometrical configuration (*viz* $z{\cdot}z$) is already there – only the number of times it is present has to be calculated.

What the text is aiming at is thus a problem of the type "square area minus sides equals number" (minus, because the 28 surfaces correspond to a rectangle which is 5′ longer than it should be). So far, we have obtained

$$28\cdot z^2 - n\cdot z = 10'.$$

This will have to be brought to normal form, for which reason it is raised to 28, the "surfaces":

$$(28z)^2 - n\cdot(28z) = 28\cdot 10' = 4°40', \qquad (**)$$

cf. Figure 3A ($S = 28z$). The next step (Figure 3, B–C) consists in "breaking" (bisecting) the rectangular area representing the number of sides $n\cdot 28z$ which is represented by a breaking of n, and in moving it around so as to transform the rectangle into a gnomon and thereby allow a quadratic completion. For this purpose, n has to be found. Since the excess of 28 ($z.z$) over the original rectangle has the sides $4z$ and 5′, its area is $4\cdot 5'\cdot z = 20'\cdot z$; i.e., $n = 20'$ (line 6). 20′ is therefore bisected, "made surround" so as to yield $^n/_2\cdot{}^n/_2 = (10')^2 = 1'40''$ (line 7). Appending this quadratic surface to the gnomon completes the square on $S - {}^n/_2$ as $4°41'40'' = (2°10')^2$. Appending again 10′ where it was broken off gives $C = 2°10' + 10' = 2°20'$ (Figure 3D; line 8).

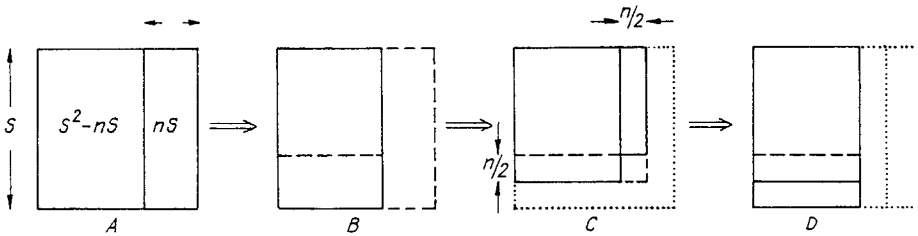


Fig. 3

This is in all respects the standard way to solve a normalized problem "square area minus sides". Next (lines 8–9), z is found as $S/28$ (a division which, again, is performed as all divisions by sexagesimally irregular numbers). x' is found as $7.z$, and the original length x by tearing out the "tearing-out of the length".

It may be of some interest to compare the path which is actually followed to an alternative which was within reach but not used. For brevity, I describe it in algebraic symbols. The original problem is

$$y\cdot({}^7/_4 y - 5') = 10'$$

or

$$1°45'\cdot y^2 - 5'\cdot y = 10'.$$

This could be normalized through multiplication by 1°45′ into

$$(1°45'\cdot y)^2 - 5\cdot(1°45'\cdot y) = 17'30''$$

and then solved as above. In principle this is equivalent to the actual method, since 1°45′ is just as good a number as 28. Furthermore, it bypasses the apparent detour over an intermediate variable z ("the fourth of the width"). That this structurally simpler method is none the less avoided demonstrates that the text, though not provided with a separate didactical explanation, is still meant to function at a level where insight in the meaning of what goes on is important (the alternative way might be followed at the level where training of methods beyond the intuitively meaningful was possible and aimed at–most of the mathematical "series texts" belong here). A rectangle containing 7 times 4 small squares is, after all, easier to grasp visually than one containing 1,45 times 1 (whether this be understood in the proper order of magnitude, as $1\,{}^3/_4$ times 1, or as 105 times 60).

This appeal to visual insight is made more explicit in problem B, which is otherwise a close parallel to problem A. This time, we only go once with the 4th of the width, which makes us fall short of the length by 5′.[16] The area is still 10′. In symbols thus

$$y + 1\cdot{}^1/_4 y = x - 5' \qquad x\cdot y = 10'.$$

Once again, the fourth of the width (z) is taken as the side of an auxiliary square. In spite of some missing signs in the beginning of line 13 the "counterpart 4" seems to tell us that a square of 4 (fourths of the width, as in line 3) times its counterpart 4 is drawn (see Figure 4) and "1 the going" then appended to the length, giving a rectangle of 5.4 small squares, which fills the original rectangle apart from a strip of 5′ times $4z$.

This time, the problem in z is thus of the type "square areas plus sides equals number". *Mutatis mutandis,* everything runs as before from this point onwards.[17]

[16] That this must be the meaning of the final part of line 11 is clear from the following. On the whole, the signs to be read on the autography make no obvious sense, even though it might be attractive to see some of them (sag *i-ši-ma*) as indications that the text introduces a trivial complication, *viz* the width and and ¼ of the width, raised to the width, falls short of the surface 10′ by 5′ widths. Still, a grammatical form *i-ši* seems quite out of place.

[17] Since the number of small squares is now regular (*viz* 20), we must presume the division to be made through raising to the igi. This restitution will also fit into the broken part of line 18, and it is indeed suggested in the autography in TMS. None the less, E. M. Bruins claims in the commentary (p. 62) that „le scribe se demande de nouveau: par quoi faut-il multiplier 20° pour obtenir 1° 40′ et il voit que c'est 5′". Dividing in this manner by a number belonging in the standard table of reciprocals would be totally unprecedented. Comparison with line 8, shows moreover, that the phrase in question could not be fitted into the lacuna.

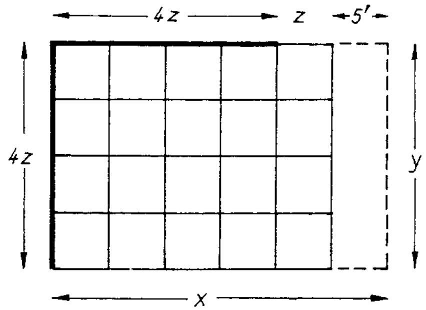


Fig. 4

Subdivision into smaller squares is appealed to in a number of other Old Babylonian mathematical texts:

First of all in BM 8390 (MKT I, 335–337; cf. Høyrup, AoF 17, 281–285, which discusses its first problem in detail). There, the use of the "multiplication" *eṣēpum* makes it clear that an integer number of squares (a concretely repeated square) is thought of, and not a mere numerical multiple; this agrees with the interpretation of the introduction of the auxiliary unit $^1/_4y$ in the present text as a means to obtain intuitive transparency.

Next also BM 13901 Nos 10 and 11 (MKT III, 2 f.; cf. Høyrup, AoF 17, 278–280, where No 10 is discussed). In those problems it is not possible to decide from the text alone whether a subdivision into smaller squares ($x \cdot x = 49 z \cdot z$) or the creation of a new reference square $7 \cdot 7$ is meant. Our Susa text supports the former interpretation, thus contributing hopefully the cover to the coffin of the myth (or, at best, the equivocal statement) that the Babylonians would use the number *1* in the function of an algebraic unknown (say, instead of a modern algebraic x).

Reanalysis of these two Susa texts have thus confirmed the conclusion which was suggested by my earlier analysis of texts IX and XVI: That the mathematical Susa texts, because of their tendency to make things explicit which are tacitly presupposed in texts from the core area, contain important clues to the methods and conceptualizations of Old Babylonian mathematics in general.